1. Coordinate Systems

a) At what times at winter solstice, summer solstice, spring equinox, and fall equinox are geocentric equatorial inertial coordinates (GEI) and geographic coordinates (GEO) approximately equal?

b) For which dates are geocentric solar ecliptic (GSE) and geocentric solar magnetospheric coordinates approximately the same? For which dates is the difference largest?

Solution:

(a) GEI is an inertial coordinate system which for all practical purposes is invariant for the astronomical background. GEO coordinates have the z axis (rotation axis) in common with GEI. At Spring equinox x_{GEI} points toward the sun, therefore the greenwich meridian x_{GEO} has to be at noon (compare Figure 1.5) for the systems to agree. At Summer solistice x_{GEI} points toward dawn, i.e., the coordinates agree at 6 am UT. Correspondingly at Fall equinox the coordinate systems agree at midnight and at Winter solistice they agree at 6 pm UT.

(b) The answer is only approximate because there is a diurnal variation of the GSM coordinates since the rotation axis and magnetic dipole axis are offset. At the solistices the rotation axis is in the plane determined by x_{GSE} (toward the sun) and z_{GSE} (normal to ecliptic). Therefore the component of the rotation axis perpendicular to x_{GSE} is identical to the normal of the ecliptic. However, because of the diurnal variation of the magnetic north direction this occurs only twice a day during a time period around the solistices. The largest deviation between the coordinate systems occurs during the equinoxes. At that time the rotation axis is perpendicular to the sunward x direction and deviates by 23 degrees from the normal of the ecliptic. Magnetic north again has a diurnal variation of 11 degrees such that the maximum deviation between GSE and GSM is a rotation of about 34 degrees of the z and y axes at one time during the day at the equinoxes.

2. Conservative and Non-Conservative Forces

A force is give by the components $\mathbf{K} = (2xy, x^2, x)$. Choose a circle with radius 1 in the x, y plane and demonstrate that the value of the closed line integral $\oint_C \mathbf{K} \cdot d\mathbf{s}$ along this circle is 0. Does this prove that the force is conservative? Apply the two other test for conservative forces to demonstrate the $\mathbf{K}$ is not conservative.

Solution:

Parameterization of the unit circle: $x = \cos\phi$, $y = \sin\phi$. Thus the integral is

$$
\begin{aligned}
\oint_C \mathbf{K} \cdot d\mathbf{s} &= \int_0^{2\pi} \left[K_x dx\,(\phi) + K_y dy\,(\phi)\right] \\
&= \int_0^{2\pi} \left[2\cos\phi \sin\phi d\,(\cos\phi) + \cos^2\phi d\,(\sin\phi)\right] \\
&= \int_0^{2\pi} \left[-2\cos\phi \sin^2\phi d\phi + \cos^3\phi d\phi\right] = 0
\end{aligned}
$$

No, this is a necessary condition but not sufficient because any closed line integral must be 0 for a potential field!

Derivation of a potential $\mathbf{K} = -\nabla\Phi$=>

$$\begin{aligned}\Phi &= -\int_x K_x dx = x^2 y + f_1(y,z)\\ \Phi &= -\int_y K_y dy = x^2 y + f_2(x,z)\\ \Phi &= -\int_z K_z dz = xz + f_3(x,y)\end{aligned}$$

Comparison of results 1 (f_1) and 2 ((f_2) imply through comparison that f_1 and f_2 can only depend on z but this contradicts result 3. Another way to look at it: One can combine the results to yield $\Phi = x^2 y + xz$ which is consitent with 2 and 3 but contradicts 1. Therfore there is no potential and the force is not conservative

Test of $\nabla \times \mathbf{K}$:

$$\nabla \times \mathbf{K} = \begin{pmatrix} \partial_y K_z - \partial_z K_y \\ \partial_z K_x - \partial_x K_z \\ \partial_x K_y - \partial_y K_x \end{pmatrix} = \begin{pmatrix} 0 - 0 \\ -1 \\ 2x - 2x \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

Since $\nabla \times \mathbf{K} \neq (0,0,0)$ The force is not conservative!

3. Quasi-Neutrality

(a) Demonstrate that the plasma definition $\langle e\phi \rangle \ll \left\langle \frac{m}{2} v^2 \right\rangle = k_B T$ implies quasineutrality, i.e., $\Lambda = n\lambda_D^3 \gg 1$

(b) The plasma parameter has the basic dependence$\Lambda \propto n_0^{-1/2} T^{3/2}$. While the dependence on temperature is intuitively clear the density dependence appears odd because lower densities mean less particles and less shielding. Why does the plasma parameter improve (increase) with decreasing density?

Solution:

(a) With $\phi_{typ} \simeq e/(\epsilon_0 r_{typ})$ and $n \simeq 1/r_{typ}^3$

$$\Lambda = n\left(\frac{\epsilon_0 k_B T}{ne^2}\right)^{3/2} = n^{-1/2}\left(\frac{\epsilon_0 k_B T}{e^2}\right)^{3/2} \simeq r_{typ}^{3/2}\left(\frac{\epsilon_0 \left\langle \frac{m}{2} v^2 \right\rangle}{e^2}\right)^{3/2} \simeq \left(\frac{\left\langle \frac{m}{2} v^2 \right\rangle}{\langle e\phi \rangle}\right)^{3/2} \gg 1$$

(b) Because of $n \simeq 1/r_{typ}^3$ decreasing density means larger typical separation of particles. However for a larger separation the potential energy becomes smaller ($\sim 1/r$) such that the plasma approximation improves for constant average kinetic energy.

4. Moments of a distribution function

A Maxwellian velocity distribution function is given by

$$f(\mathbf{v}) = n\left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{m}{2k_B T}\left[v_x^2 + v_y^2 + (v_z - v_{z0})^2\right]\right)$$

Compute the average velocity of particles described by the distribution function. The kinetic energy can be split into a thermal portion and a part which is caused by the bulk motion of particles. Compute the thermal kinetic energy? What is the kinetic energy from the bulk motion.

Solution:

Average Velocity:

$$\mathbf{u} = \frac{1}{n}\int\limits_v d^3v\mathbf{v}f(\mathbf{v})$$

Since the distribution is symmetric for the v_x and v_y components the corresponding integrals are 0. For the z component we obtain by substituting $w_z = v_z - v_{z0}$

$$\begin{aligned} u_z &= \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int\limits_v d^3v v_z \exp\left(-\frac{m}{2k_B T}\left[v_x^2 + v_y^2 + (v_z - v_{z0})^2\right]\right) \\ &= \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int\limits_v dv_x dv_y dw_z (w_z + v_{z0}) \exp\left(-\frac{m}{2k_B T}\left[v_x^2 + v_y^2 + w_z^2\right]\right) \\ &= v_{z0}\left(\frac{m}{2\pi k_B T}\right)^{3/2} \int\limits_v dv_x dv_y dw_z \exp\left(-\frac{m}{2k_B T}\left[v_x^2 + v_y^2 + w_z^2\right]\right) \end{aligned}$$

Here

$$\left(\frac{m}{2\pi k_B T}\right)^{3/2} \int\limits_v dv_x dv_y dw_z \exp\left(-\frac{m}{2k_B T}\left[v_x^2 + v_y^2 + w_z^2\right]\right) = 1$$

by definition because multiplyng the above equation with n is the definition of the plasma density such that

$$u_z = v_{z0}$$

Another way to obtain this result is to realize that the given distribution function is a Maxwell distribution transformed into the frame moving with v_{z0}. Since the average velocity of a Maxwell distribtion is 0 the average velocity of the transformed distribution must be v_{z0}.

Total kinetic energy:

$$E_{kin,tot} = \int\limits_v d^3v \frac{m}{2}\left(v_x^2 + v_y^2 + v_z^2\right) f(\mathbf{v})$$

With the transformation $w_z = v_z - v_{z0}$this becomes:

$$E_{kin,tot} = \frac{m}{2}n\left(\frac{m}{2\pi k_B T}\right)^{3/2} \int\limits_v dv_x dv_y dw_z \left(v_x^2 + v_y^2 + (w_z + v_{z0})^2\right) \exp\left(-\frac{m}{2k_B T}\left[v_x^2 + v_y^2 + w_z^2\right]\right)$$

$$
\begin{aligned}
&= \frac{m}{2} n \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_v dv_x dv_y dw_z \left(v_x^2 + v_y^2 + w_z^2\right) \exp\left(-\frac{m}{2k_B T}\left[v_x^2 + v_y^2 + w_z^2\right]\right) \\
&\quad + 2v_{z0} \frac{m}{2} n \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_v dv_x dv_y dw_z w_z \exp\left(-\frac{m}{2k_B T}\left[v_x^2 + v_y^2 + w_z^2\right]\right) \\
&\quad + \frac{m}{2} n v_{z0}^2 \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_v dv_x dv_y dw_z \exp\left(-\frac{m}{2k_B T}\left[v_x^2 + v_y^2 + w_z^2\right]\right)
\end{aligned}
$$

Here the first term is the kinetic energy in restframe of the Maxwellian or the thermal kinetic energy, the second term is 0 because of anti-symmetry in w_z (note the integral has the bounds $-\infty \leq w_z \leq \infty$) and the last term represents the bulk kinetic energy. Because of the normalization of the integral the bulk kinetic energy is simply

$$
E_{bulk} = \frac{m}{2} n v_{z0}^2 = \frac{1}{2} \rho u_z^2
$$

The thermal kinetic energy is

$$
\begin{aligned}
E_{th} &= \frac{m}{2} n \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_v dv_x dv_y dw_z \left(v_x^2 + v_y^2 + w_z^2\right) \exp\left(-\frac{m}{2k_B T}\left[v_x^2 + v_y^2 + w_z^2\right]\right) \\
&= \frac{3m}{2} n \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_v dv_x dv_y dw_z v_x^2 \exp\left(-\frac{m}{2k_B T}\left[v_x^2 + v_y^2 + w_z^2\right]\right) \qquad (1)
\end{aligned}
$$

because the v_x^2 term, the v_y^2 term, and the w_z^2 term have identical contributions to the energy. Substituting $v_x = \sqrt{k_B T/m}\,\widetilde{v}_x$, and the same for v_y and w_z yields

$$
\begin{aligned}
E_{th} &= \frac{3m}{2} n \left(\frac{1}{2\pi}\right)^{3/2} \frac{k_B T}{m} \int_v d\widetilde{v}_x d\widetilde{v}_y d\widetilde{w}_z \widetilde{v}_x^2 \exp\left(-\frac{1}{2}\left[\widetilde{v}_x^2 + \widetilde{v}_y^2 + \widetilde{w}_z^2\right]\right) \\
&= \frac{3}{2} n k_B T \left(\frac{1}{2\pi}\right)^{3/2} \int_{-\infty}^{\infty} d\widetilde{v}_x \widetilde{v}_x^2 \exp\left(-\frac{1}{2}\widetilde{v}_x^2\right) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\widetilde{v}_y d\widetilde{w}_z \exp\left(-\frac{1}{2}\left[\widetilde{v}_y^2 + \widetilde{w}_z^2\right]\right)
\end{aligned}
$$

Here $\widetilde{v}_y$ and $\widetilde{w}_z$ are replaced by polar coordinates with $v_r^2 = \widetilde{v}_y^2 + \widetilde{w}_z^2$:

$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\widetilde{v}_y d\widetilde{w}_z \exp\left(-\frac{1}{2}\left[\widetilde{v}_y^2 + \widetilde{w}_z^2\right]\right) &= \left[\int_{-\infty}^{\infty} dx \exp\left(-\frac{1}{2}x^2\right)\right]^2 \qquad (2) \\
&= \int_0^{\infty} dv_r 2\pi v_r \exp\left(-\frac{1}{2}v_r^2\right) \\
&= 2\pi \left[-\exp\left(-\frac{1}{2}v_r^2\right)\right]_0^{\infty} = 2\pi
\end{aligned}
$$

Thus the thermal energy becomes

$$
E_{th} = \frac{3}{2} n k_B T \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} d\widetilde{v}_x \widetilde{v}_x^2 \exp\left(-\frac{1}{2}\widetilde{v}_x^2\right)
$$

We can use integration by parts and the expression in (2) or alternatively we can use the following integral identities directly in (1)

$$
\begin{aligned}
K_0 &= \int_{-\infty}^{+\infty} \exp\left(-a^2u^2\right) du = \frac{\sqrt{\pi}}{a} \\
K_2 &= \int_{-\infty}^{+\infty} u^2 \exp\left(-a^2u^2\right) du = \frac{\sqrt{\pi}}{2a^3}
\end{aligned}
$$

to obtain the thermal energy density as

$$
E_{th} = \frac{3}{2} n k_B T
$$