5. Plasma approximations

a) Assume a plasma density of 1 cm^{-3}, temperature equivalent to 1 keV, and a magnetic field of 20 nT which are typical for the near Earth magnetotail. Determine Debye radius, and the electron and ion inertia scales. Can one neglect the electron inertial, and the Hall terms in Ohm's law for these parameters if a typical length scale of the tail current sheet is $4\, R_E$(1 R_E=6400 km)?

b) Compute the plasma parameter, the electron plasma frequency, the collision frequency for electrons, and the thermal velocity, and the mean free path of an electron based on these numbers. Can one neglect collisions for these electrons for the length scales of the magnetosphere?

Solution:

(a)

- Debye radius $\lambda_D = 7.43 \cdot 10^2 T_e^{1/2}/n^{1/2} cm = 2.35 \cdot 10^4 cm$ = 235m
- Electron inertia scale $\lambda_e = 5.31 \cdot 10^5 n^{-1/2} cm = 5.31 \cdot 10^5 cm$ = 5.31km
- Ion inertia scales $\lambda_i = (m_i/m_e)^{1/2} \lambda_e = 2.3 \cdot 10^7 cm$ = 230km
- 4 $R_E = 25800\, km \gg \lambda_i \gg \lambda_e$; therefore electron and ion inertial terms can be negelcted in Ohm's law

(b)

- Plasma parameter $\Lambda_D = 4.10 \cdot 10^8 T_e^{3/2} n^{-1/2} = 1.30 \cdot 10^{13}$
- Plasma frequency: $\omega_{pe} = 5.64 \cdot 10^4 n^{1/2} = 5.64 \cdot 10^4\, s^{-1}$
- Collision frequency: $\nu_e = 1.37 \cdot 10^{-6} n T_e^{-3/2} \ln \Lambda_D\, s^{-1} = 4.33 \cdot 10^{-11} \ln \Lambda_D\, s^{-1} = 1.31 \cdot 10^{-9}\, s^{-1}$
- Thermal velocity: $v_{the} = 4.19 \cdot 10^7 T_e^{1/2} = 1.3 \cdot 10^9\, cm/s = 1.3 \cdot 10^4 km/s$
- Mean free path: $L_c = v_{the}/\nu_e = 1.0 \cdot 10^{13} km = 1.6 \cdot 10^9 R_E$

The mean free path is many order of magnitude larger than any dimension associated with the Earth's magnetosphere. Therfore the system is highly collisionless.

6. Plasma properties

For the plasma in above problem, determine the temperature in degrees Kelvin. Determine the energy density in kW hours/m^3 and kW hours /R_E^3. For the sake of simplicity assume that the plasma sheet is represented by a cylinder with 10 R_E radius and 100 R_E length. How long could a power plant with an output of 1000 MW operate on the thermal energy stored in the plasma?

Solution:

- Temperature in degrees Kelvin: $T = 1.16 \cdot 10^7 K$
- Energy density: $E_{th} = \frac{3}{2} n k_B T$

$$\begin{aligned} \varepsilon &= 1.5 \cdot 10^6 \cdot 1.38 \cdot 10^{-23} \cdot 1.16 \cdot 10^7 \mathrm{Jm}^{-3} = 2.4 \cdot 10^{-10} \mathrm{J/m}^3 \\ &= \frac{1.6 \cdot 10^{-10}}{10^3 \cdot 3600} \mathrm{kWh/m}^3 = 6.7 \cdot 10^{-17} \mathrm{kWh/m}^3 \end{aligned}$$

- In kW hours/R_E^3: $\varepsilon = 6.7 \cdot 10^{-17} \cdot 6.4^3 \cdot 10^{18} = 1.7 \cdot 10^4 \mathrm{kWh}\ /\mathrm{R_E^3}$.
- Energy in 10 R_E radius and 100 R_E length cylinder: $W = \pi \cdot 10^4 \cdot 1.7 \cdot 10^4 = 5.5 \cdot 10^8 \mathrm{kWh}$
- Operation time for 1000 MW power plant $t = W/10^6 = 550$ hours$\approx$ 23 days

7. Collisionless Boltzmann equation

Show that any distribution function $f(\mathbf{x}, \mathbf{v}, t) = F(H)$ with $H = m\mathbf{v}^2/2 + q\phi$ solves the steady state ($\partial/\partial t = 0$) collisionless Boltzmann equation.

Solution:

Collisionless Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f = 0$$

Individual terms:

$$\mathbf{v} \cdot \nabla f = \mathbf{v} \cdot \frac{dF}{dH} \frac{dH}{d\phi} \nabla \phi = -q \frac{dF}{dH} \mathbf{v} \cdot \mathbf{E}$$

and

$$\begin{aligned} \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f &= \frac{q}{m} \frac{dF}{dH} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} H \\ &= m \frac{q}{m} \frac{dF}{dH} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} \\ &= q \frac{dF}{dH} \mathbf{E} \cdot \mathbf{v} \end{aligned}$$

such that the sum of the two terms is zero and $\partial f / \partial t = 0$

8. Fluid equations

For the case of isotropic pressure, no heatflux $\mathbf{L} = 0$, and no resitivity $\eta = 0$, demonstrate that the function $h = p^{1/\gamma}$ satisfies a continuity equation, i.e., $\partial h/\partial t + \nabla \cdot h\mathbf{u} = 0$.

Solution:

With $p = h^\gamma$ substituted into the isotropic pressure equation

$$\frac{\partial p}{\partial t} + \nabla \cdot p\mathbf{u} = -\left(\gamma - 1\right) p \nabla \cdot \mathbf{u}$$

we obtain

$$\begin{aligned}
\frac{\partial h^\gamma}{\partial t} = \gamma h^{\gamma-1}\frac{\partial h}{\partial t} &= -\nabla \cdot h^\gamma \mathbf{u} - \left(\gamma - 1\right) h^\gamma \nabla \cdot \mathbf{u} \\
&= -h^\gamma \nabla \cdot \mathbf{u} - \gamma h^{\gamma-1} \mathbf{u} \cdot \nabla h - \left(\gamma - 1\right) h^\gamma \nabla \cdot \mathbf{u} \\
&= -\gamma h^{\gamma-1} \mathbf{u} \cdot \nabla h - \gamma h^\gamma \nabla \cdot \mathbf{u}
\end{aligned}$$

Division by $h^{\gamma-1}$ yields

$$\begin{aligned}
\gamma \frac{\partial h}{\partial t} &= -\gamma \mathbf{u} \cdot \nabla h - \gamma h \nabla \cdot \mathbf{u} \\
&= -\gamma \left(\mathbf{u} \cdot \nabla h + h \nabla \cdot \mathbf{u}\right)
\end{aligned}$$

Division by γ thus yields

$$\frac{\partial h}{\partial t} + \nabla \cdot h\mathbf{u} = 0$$