12. Dipole magnetic field

a) Assume the magnetic field of the Earth to be dipolar. Two magnetic field lines are radially separated by 1000 km in the magnetic equator at a distance of 5 Earth radii (5 R_E). What is the separation at the Earth's surface?

b) Consider the superposition of a constant IMF $\mathbf{B}_{IMF} = -B_0 \mathbf{e}_z$ to the dipole field of the Earth. At what radial distance is the X-line that separates open and closed field for $B_0 = 3 \cdot 10^{-9} T$ and $B_0 = 3 \cdot 10^{-8} T$. Estimate the latitude on the Earth's surface (polar cap) from which the magnetic field is open into space for the two IMF values. (Assume as illustrated in class that the IMF and the Earth's dipole field can be linearly superimposed. Hint: Use the field line equation to map the magnetic field lines connected to the X-line to the Earth's surface.)

Solution:

a) Fieldline equation: $\tilde{r} = L \cos^2 \lambda$

Latitude at the Earth's surface: $\lambda = \arccos \sqrt{1/L}$

For $L = 5$: $63.435°$

For $L = 5 + 1000/6400 = 5.156$: $63.872°$

This corresponds to a distance of about 48.8 km or 30 miles (Note $1°$ corresponds to about 111.7 km).

b) Estimate the latitude on the Earth's surface from which the magnetic field is open into space for $\mathbf{B}_{IMF} = -B_0 \mathbf{e}_z$ with $B_0 > 0$.

Equatorial magnetic field: $B_{eq} = -B_0 + B_E R_E^3 / r^3$ with $B_E = 3.11 \cdot 10^{-5} T$

X line, $B_{eq} = 0$ => $r_{xl} = (B_E/B_0)^{1/3} R_E$ or for L-shell: $L_{xl} = (B_E/B_0)^{1/3}$

For $B_0 = 3 \cdot 10^{-9} T$: $r_{xl,1} = 21.8\, R_E$

For $B_0 = 3 \cdot 10^{-8} T$: $r_{xl,1} = 10.1\, R_E$

Field line equation:

$$r^2 \cos^2 \lambda \left(-\frac{B_0}{2} - \frac{B_E R_E^3}{r^3} \right) = const = r_{xl}^2 \left(-\frac{B_0}{2} - \frac{B_E R_E^3}{r_{xl}^3} \right)$$

On Earth's surface $r = 1\, R_E$:

$$\begin{aligned} \cos^2 \lambda_E \left(\frac{B_0}{2} + B_E \right) &= L_{xl}^2 \left(\frac{B_0}{2} + \frac{B_E}{L_{xl}^3} \right) \\ &= \left(\frac{B_E}{B_0} \right)^{2/3} \left(\frac{B_0}{2} + B_0 \right) \end{aligned}$$

or

$$\cos^2 \lambda_E = \left(\frac{B_E}{B_0} \right)^{2/3} \frac{3B_0}{B_0 + 2B_E} = 3 \left(\frac{B_E}{B_0} \right)^{2/3} \frac{1}{1 + 2B_E/B_0}$$

For $B_E \gg B_0$:

$$\cos^2 \lambda_E \simeq 3 \left(\frac{B_E}{B_0}\right)^{2/3} \frac{B_0}{2B_E} = \frac{3}{2} \left(\frac{B_0}{B_E}\right)^{1/3}$$

and

$$\lambda_E \simeq \arccos \left[\frac{3}{2} \left(\frac{B_0}{B_E}\right)^{1/3}\right]^{1/2} = \arccos \left[\frac{3}{2L_{xl}}\right]^{1/2}$$

For $B_0 = 3 \cdot 10^{-9} T$: $\lambda_e = 74.79°$

For $B_0 = 3 \cdot 10^{-8} T$: $\lambda_e = 67.36°$

13. Current layer magnetic field

Consider a magnetic field given by $\mathbf{B} = y\mathbf{e}_x + \alpha \sin(kx)\, \mathbf{e}_y$ and assume $\alpha, k > 0$.

a) Derive the equations for the field lines.

b) Determine the vector potential and the two Euler potentials.

c) Sketch and discuss the field lines.

d) Determine the condition for α and k such that the the current density at X lines is 0.

Solution:

a) Derive the equations for the field lines.

$$B_x = \partial_y A_z = y \quad => \quad A_z = \frac{1}{2} y^2 + f(x)$$
$$B_y = -\partial_x A_z = \alpha \sin(kx) \quad => \quad A_z = \frac{\alpha}{k} \cos(kx) + g(y)$$

Combination of the two yields

$$A_z = \frac{1}{2} y^2 + \frac{\alpha}{k} \cos(kx)$$

Fields lines $A_z = const$: $\frac{1}{2} y^2 + \frac{\alpha}{k} \cos(kx) = C$

b) Determine the vector potential and the two Euler potentials.

The vector potential is already computed. The Euler potentials are
from $\mathbf{B} = \nabla\alpha \times \nabla\beta$. Comparison with the vector potential $\mathbf{B} = \nabla A \times \mathbf{e}_z$ yields

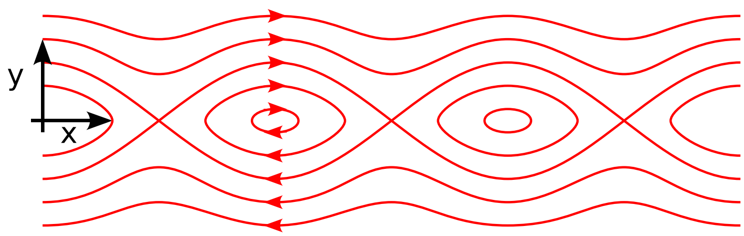
$$\nabla\beta = \mathbf{e}_z \quad or \quad \frac{\partial\beta}{\partial z} = 1$$

which yields

$$\alpha = A_z$$
$$\beta = z$$

c) Sketch and discuss the field lines.

Sketch:



For $C \geq \frac{\alpha}{k}$ field lines have maxima and minima for $\sin kx = 0$. Maxima are for $x = 2n\pi/k$, minima for $x = (2n+1)\pi/k$ (for the upper branch $y = +\sqrt{2\left(C - \frac{\alpha}{k}\cos(kx)\right)}$ and opposite for the lower branch). Field lines are periodic in x with the period $2\pi/k$.

Expansion for locations with $\mathbf{B} = 0$:

i. $C = \frac{\alpha}{k}$, $x = 0$:

$\frac{1}{2}y^2 + \frac{\alpha}{k}(1 - k^2x^2) = \frac{\alpha}{k}$ => $y = \pm\alpha k x$ **Magnetic X lines**

ii. $C < \frac{\alpha}{k}$, $x = \pi + \tilde{x}$:

$\frac{1}{2}y^2 - \frac{\alpha}{k}(1 - k^2\tilde{x}^2) = C$ => $y^2 + 2\alpha k\tilde{x}^2 = C$ **Ellipsoids - Magnetic O lines**

d) Determine the condition for α and k such that the the current density at X lines is 0.

$J_z = \frac{1}{\mu_0}(\partial_x B_y - \partial_y B_x) = \frac{1}{\mu_0}(\alpha k - 1)$

=> $J_z = 0$ for $\alpha = 1/k$

14. Local expansion of B

Consider the matrix

$$\nabla\mathbf{B} = \begin{pmatrix} \alpha_x & \gamma & \delta_x \\ \gamma' & \alpha_y & \delta_y \\ \beta_x & \beta_y & \alpha_z \end{pmatrix} \quad (1)$$

with constant parameters α_i, β_i, δ_i, γ, and γ' for the expansion of the magnetic field at the origin.

a) Assume the magnetic field at the origin $\mathbf{r} = 0$ has only a B_z component B_0. Determine the magnetic field components B_x, B_y, and B_z in the vicinity of the origin to first order in x, y, and z.

b) Derive the most general form of the matrix elements for a static vacuum magnetic field $\mathbf{j} = 0$ consistent with $\nabla \cdot \mathbf{B} = 0$.

c) Using the above result, demonstrate that the presence of nonzero curvature in a static vacuum field always implies the presence of a nonzero magnetic gradient and vice versa.

Solution:

Consider the matrix

$$\nabla\mathbf{B} = \begin{pmatrix} \alpha_x & \gamma & \delta_x \\ \gamma' & \alpha_y & \delta_y \\ \beta_x & \beta_y & \alpha_z \end{pmatrix} \quad (2)$$

with constant parameters α_i, β_i, δ_i, γ, and γ' for the expansion of the magnetic field at the origin.

a) At the origin $\mathbf{r} = 0$ has only a B_z component B_0. Magnetic field in the vicinity of the origin to first order in x, y, and z.

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} \alpha_x x + \gamma' y + \beta_x z \\ \gamma x + \alpha_y y + \beta_y z \\ B_0 + \delta_x x + \delta_y y + \alpha_z z \end{pmatrix}$$

b) $\nabla \cdot \mathbf{B} = 0$:

=> $\alpha_x + \alpha_y + \alpha_z = 0$ => $\alpha_z = -\alpha_x - \alpha_y$

$\mathbf{j} = \nabla \times \mathbf{B} = 0$

x Component: $\delta_y - \beta_y = 0$ => $\delta_y = \beta_y$

y Component: $\beta_x - \delta_x = 0$ => $\delta_x = \beta_x$

z Component: $\gamma - \gamma' = 0$

Most general form

$$\nabla \mathbf{B} = \begin{pmatrix} \alpha_x & \gamma & \beta_x \\ \gamma & \alpha_y & \beta_y \\ \beta_x & \beta_y & -\alpha_x - \alpha_y \end{pmatrix}$$

c) Demonstrate that the presence of nonzero curvature in a static vacuum field always implies the presence of a nonzero magnetic gradient and vice versa.

Nonzero curvature implies either β_x or β_y nonzero. However, implies that that either δ_x or δ_y is nonzero.