

1. General Ohm's Law:

Use the normalization to typical values to obtain the coefficients for the inertia term and for the Hall term in Ohm's law and show that these scale with the electron inertia and the ion inertia length respectively.

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{m_e m_i}{e^2 \rho} \left[\frac{\partial \mathbf{j}}{\partial t} + \nabla \cdot (\mathbf{u} \mathbf{j} + \mathbf{j} \mathbf{u}) \right] - \frac{M}{e \rho} \nabla p_e + \frac{m_i}{e \rho} \mathbf{j} \times \mathbf{B} + \eta \mathbf{j} \quad (1)$$

Measuring all quantities in typical units, i.e., the magnetic induction \mathbf{B} in units of a typical magnetic field B_0 such that $\mathbf{B} = B_0 \hat{\mathbf{B}}$ where $\hat{\mathbf{B}}$ is now of order unity, we can examine the coefficients of the different terms in Ohm's law. Note that velocities should be measured in units of the Alfvén speed $u_0 = B_0 / \sqrt{\mu_0 \rho_0}$, time in units of Alfvén travel time $t_0 = L_0 / u_0$, and length scales in a typical scale for gradients in the system L_0 . The normalization for current density is obtained from Ampere's law

$$\frac{B_0}{L_0} \hat{\nabla} \times \hat{\mathbf{B}} = \mu_0 j_0 \hat{\mathbf{j}}$$

which yields

$$j_0 = \frac{B_0}{\mu_0 L_0}$$

In Ohm's law one obtains

$$E_0 \mathbf{E} + u_0 B_0 \mathbf{u} \times \mathbf{B} = \frac{j_0}{t_0 m_i n_0} \frac{m_e m_i}{e^2 \hat{\rho}} \left[\frac{\partial \hat{\mathbf{j}}}{\partial \hat{t}} + \hat{\nabla} \cdot (\hat{\mathbf{u}} \hat{\mathbf{j}} + \hat{\mathbf{j}} \hat{\mathbf{u}}) \right] + \frac{j_0 B_0}{m_i n_0} \frac{m_i}{e \hat{\rho}} \hat{\mathbf{j}} \times \hat{\mathbf{B}} + \dots$$

This shows that $E_0 = u_0 B_0$ and dividing Ohm's law by $u_0 B_0$ yields the coefficient for the inertia term as

$$\begin{aligned} \frac{j_0}{t_0 m_i n_0} \frac{1}{u_0 B_0} \frac{m_e m_i}{e^2} &= \frac{B_0}{\mu_0 L_0} \frac{u_0}{L_0 m_i n_0} \frac{1}{u_0 B_0} \frac{m_e m_i}{e^2} \\ &= \frac{1}{L_0^2} \frac{1}{\epsilon_0 \mu_0} \frac{\epsilon_0 m_e}{n_0 e^2} \\ &= \frac{c^2}{L_0^2 \omega_{pe}^2} \end{aligned}$$

For the Hall term we obtain:

$$\begin{aligned} \frac{j_0 B_0}{m_i n_0} \frac{1}{u_0 B_0} \frac{m_i}{e} &= \frac{B_0}{\mu_0 L_0} \frac{B_0}{m_i n_0} \frac{\sqrt{\mu_0 m_i n_0}}{B_0^2} \frac{m_i}{e} \\ &= \frac{1}{L_0} \frac{1}{\sqrt{\epsilon_0 \mu_0}} \frac{\sqrt{\epsilon_0 m_i}}{\sqrt{n_0 e^2}} \\ &= \frac{c}{L_0 \omega_{pi}} \end{aligned}$$

such that the normalized version of Ohms law becomes

$$E_0 \mathbf{E} + u_0 B_0 \mathbf{u} \times \mathbf{B} = \frac{c^2}{L_0^2 \omega_{pe}^2} \frac{1}{\hat{\rho}} \left[\frac{\partial \hat{\mathbf{j}}}{\partial \hat{t}} + \widehat{\nabla} \cdot (\hat{\mathbf{u}} \hat{\mathbf{j}} + \hat{\mathbf{j}} \hat{\mathbf{u}}) \right] + \frac{c}{L_0 \omega_{pi}} \frac{1}{\hat{\rho}} \hat{\mathbf{j}} \times \hat{\mathbf{B}} + \dots$$

here c/ω_{pe} is the electron inertia length and c/ω_{pi} such that the electron inertia term scales with the square of the electron inertia length. In other words if the typical length scale of plasma gradients is comparable to the electron inertia scale the inertia term must be considered. It can be neglected if gradient scales are much larger than the electron inertia scale.

2. Moments of distribution functions:

a) **Can a single Maxwellian distribution cause a nonzero heatflux.** Explain your answer.

$$\mathbf{L}(\mathbf{x}, t) = \frac{1}{2} m \int_{-\infty}^{\infty} d^3v (\mathbf{v} - \mathbf{u})(\mathbf{v} - \mathbf{u})^2 f(\mathbf{x}, \mathbf{v}, t)$$

No, single Maxwellian cannot generate heatflux. Assume the distribution in the restframe such that heatflux for instance along x is the integral over $v_x v^2 f(v)$. The integral is always 0 because of the symmetry of the v_x integration.

Consider a distribution function which consists of a superposition of two Maxwell distributions:

$$f(\mathbf{v}) = n_0 \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp\left(-\frac{mv^2}{2k_B T}\right) + n_1 \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp\left(-\frac{m[(v_x - u_1)^2 + v_y^2 + v_z^2]}{2k_B T}\right)$$

b) **Number density,**

$$\begin{aligned} n &= \int_{-\infty}^{\infty} d^3v f(v) \\ &= \int_{-\infty}^{\infty} d^3v f_0(v) + \int_{-\infty}^{\infty} d^3v f_1(v) \end{aligned}$$

with

$$\begin{aligned} f_0(\mathbf{v}) &= n_0 \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp\left(-\frac{mv^2}{2k_B T}\right) \\ f_1(\mathbf{v}) &= n_1 \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp\left(-\frac{m[(v_x - u_1)^2 + v_y^2 + v_z^2]}{2k_B T}\right) \end{aligned}$$

Each of these functions is a Maxwell distribution (the shift in velocity for f_1 can be removed for the density integration with the substitution $\tilde{v}_x = v_x - u_1$) such that each contribution just yields the density n_0 and n_1 or

$$n = n_0 + n_1$$

c) **Bulk velocity:**

$$\mathbf{u} = \frac{1}{n} \left\{ \int_{-\infty}^{\infty} d^3v \mathbf{v} f_0(v) + \int_{-\infty}^{\infty} d^3v \mathbf{v} f_1(v) \right\}$$

The first integral yield 0 which one can illustrate by considering the x component

$$\begin{aligned} \int_{-\infty}^{\infty} d^3v v_x f_0(v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} v_x f_0(v) dv_x \right) dv_y dv_z \\ &= 0 \end{aligned}$$

because $v_x f_0(v)$ is antisymmetric and the integral from $-\infty$ to 0 cancels the integral from 0 to ∞ . The same argument applies to the u_y and the u_z components in the second integral. The u_x component is

$$\begin{aligned}
\int_{-\infty}^{\infty} d^3v v_x f_1(v) &= n_1 \left(\frac{m}{2\pi k_B T} \right)^{3/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v_x \exp \left(-\frac{m \left[(v_x - u_1)^2 + v_y^2 + v_z^2 \right]}{2k_B T} \right) dv_x dv_y dv_z \\
&= n_1 \left(\frac{m}{2\pi k_B T} \right)^{3/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\tilde{v}_x + u_1) \exp \left(-\frac{m \left[\tilde{v}_x^2 + v_y^2 + v_z^2 \right]}{2k_B T} \right) d\tilde{v}_x dv_y dv_z \\
&= u_1 n_1 \left(\frac{m}{2\pi k_B T} \right)^{3/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left(-\frac{m \left[\tilde{v}_x^2 + v_y^2 + v_z^2 \right]}{2k_B T} \right) d\tilde{v}_x dv_y dv_z \\
&= n_1 u_1
\end{aligned}$$

Thus the bulk velocity is

$$\mathbf{u} = \frac{n_1}{n} u_1 \mathbf{e}_x$$

d) and heatflux (along the x direction):

For the same symmetry reasons the heatflux in the y and z directions must be 0. For the x direction one obtains

$$\begin{aligned}
L_x &= \frac{m}{2} \int_{-\infty}^{\infty} d^3v (v_x - u_x) (\mathbf{v} - \mathbf{u})^2 f(v) \\
&= \frac{m}{2} \int_{-\infty}^{\infty} d^3v (v_x - u_x) \left[(v_x - u_x)^2 + v_y^2 + v_z^2 \right] f(v) \\
&= \frac{m}{2} \int_{-\infty}^{\infty} d^3v (v_x - u_x) \left[v_y^2 + v_z^2 \right] f(v) \\
&\quad + \frac{m}{2} \int_{-\infty}^{\infty} d^3v (v_x - u_x) (v_x - u_x)^2 f(v) \\
&= \frac{m}{2} \int_{-\infty}^{\infty} d^3v (v_x - u_x) \left[v_y^2 + v_z^2 \right] f_0(v) + \frac{m}{2} \int_{-\infty}^{\infty} d^3v (v_x - u_x) \left[v_y^2 + v_z^2 \right] f_1(v) \\
&\quad + \frac{m}{2} \int_{-\infty}^{\infty} d^3v (v_x - u_x) (v_x - u_x)^2 f_0(v) + \frac{m}{2} \int_{-\infty}^{\infty} d^3v (v_x - u_x) (v_x - u_x)^2 f_1(v)
\end{aligned}$$

Contributions from f_0

$$\begin{aligned}
L_{x0} &= \frac{m}{2} \int_{-\infty}^{\infty} d^3v (v_x - u_x) \left[v_y^2 + v_z^2 \right] f_0(v) + \frac{m}{2} \int_{-\infty}^{\infty} d^3v (v_x - u_x) (v_x - u_x)^2 f_0(v) \\
&= -\frac{m}{2} u_x \int_{-\infty}^{\infty} d^3v \left[v_y^2 + v_z^2 \right] f_0(v) + \frac{m}{2} \int_{-\infty}^{\infty} d^3v \left(v_x^3 - 3u_x v_x^2 + 3u_x^2 v_x - u_x^3 \right) f_0(v) \\
&= -\frac{1}{2} u_x (2n_0 k_B T) - 3\frac{1}{2} u_x n_0 k_B T - \frac{m}{2} u_x^3 n_0 = -\frac{1}{2} m n_0 u_x \left(5v_{th}^2 + u_x^2 \right)
\end{aligned}$$

Contributions from f_1

$$\begin{aligned}
 L_{x1} &= \frac{m}{2} \int_{-\infty}^{\infty} d^3v (v_x - u_x) [v_y^2 + v_z^2] f_1(v) + \frac{m}{2} \int_{-\infty}^{\infty} d^3v (v_x - u_x) (v_x - u_x)^2 f_1(v) \\
 &= \frac{m}{2} \int_{-\infty}^{\infty} d^3v (\tilde{v}_x + u_1 - u_x) [v_y^2 + v_z^2] f_1(v) + \frac{m}{2} \int_{-\infty}^{\infty} d^3v (\tilde{v}_x + u_1 - u_x) (\tilde{v}_x + u_1 - u_x)^2 f_1(v) \\
 &= \frac{m}{2} \tilde{u}_1 \int_{-\infty}^{\infty} d^3v [v_y^2 + v_z^2] f_1(v) + \frac{m}{2} \int_{-\infty}^{\infty} d^3v (\tilde{v}_x^3 + 3\tilde{u}_1 \tilde{v}_x^2 + 3\tilde{u}_1^2 \tilde{v}_x + \tilde{u}_1^3) f_1(v) \\
 &\quad \text{with } \tilde{u}_1 = u_1 - u_x = \frac{n_0}{n} u_1
 \end{aligned}$$

$$L_{x1} = \frac{m}{2} \tilde{u}_1 (2n_1 k_B T) + 3 \frac{m}{2} \tilde{u}_1 n_1 k_B T + \frac{m}{2} \tilde{u}_1^3 n_1 = \frac{1}{2} m n_1 \tilde{u}_1 (5v_{th}^2 + \tilde{u}_1^2)$$

Sum of the two contributions:

$$\begin{aligned}
 L_x &= -\frac{1}{2} m n_0 u_x (5v_{th}^2 + u_x^2) + \frac{1}{2} m n_1 \tilde{u}_1 (5v_{th}^2 + \tilde{u}_1^2) \\
 &= -\frac{1}{2} \frac{m n_0 n_1}{n} u_1 \left(5v_{th}^2 + \frac{n_1^2}{n^2} u_1^2 \right) + \frac{1}{2} \frac{m n_0 n_1}{n} u_1 \left(5v_{th}^2 + \frac{n_0^2}{n^2} u_1^2 \right) \\
 &= \frac{1}{2} \frac{m n_0 n_1}{n} u_1^3 \frac{n_0^2 - n_1^2}{n^2}
 \end{aligned}$$

e) How does the heatflux depend on n_1/n_0 and on the velocity u_1 . Discuss conditions for distribution functions which cause a nonzero heatflux.

Direction of the heatflux depends on the direction of u_1 .

- Direction of the heatflux depends on the ratio of the number densities
- For equal number densities the heatflux is zero
- Result indicates that a different temperature generates a heatflux even though the densities for the two distributions may be the same.

Result indicates that a distribution function has to be skewed in velocity space. Distribution which are symmetric in v_x do not cause heatflux along x .