1. Force balance equation in cylindrical coordinates:

Assume $\partial/\partial\theta = \partial/\partial z = 0$ and $B_r = 0$ and show that the force balance equation in cylindrical coordinates (r, θ, z) takes the form

$$\frac{\partial}{\partial r}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_0}\right) + \frac{B_\theta^2}{\mu_0 r} = 0$$

2. θ pinch:

Assume a constant current j_0 in the z direction in a cylindrical coordinate system. Compute the magnetic field $B_\theta(r)$ and integrate the force balance equation to show that the pressure is given by

$$p\,(r) = -\frac{\mu_0 j_0^2}{4} r^2 + p_0$$

with $p\,(r=0) = p_0$. Determine the critical radius for which the pressure decreases to 0. Apply the stability criterion to this specific configuration and determine the radius range for which instability occurs as a function of p_0.

3. Stability for the Harris sheet:

The energy principle for a displacement $\boldsymbol{\xi}$ without considering surface terms is

$$\begin{aligned} U &= \frac{1}{2}\int_V \Big[\gamma p_0 \left(\nabla\cdot\boldsymbol{\xi}\right)^2 + \frac{1}{\mu_0}\left(\nabla\times\left(\boldsymbol{\xi}\times\mathbf{B}_0\right)\right)^2 \\ &\qquad + \boldsymbol{\xi}\cdot\nabla p_0 \nabla\cdot\boldsymbol{\xi} - \frac{1}{\mu_0}\left(\boldsymbol{\xi}\times\left(\nabla\times\mathbf{B}_0\right)\right)\cdot\nabla\times\left(\boldsymbol{\xi}\times\mathbf{B}_0\right)\Big]\, d\mathbf{x} \end{aligned}$$

Determine the stability criterion for the Harris sheet equilibrium $\mathbf{B} = B\mathbf{e}_y = B_0 \tanh\frac{x}{L}\,\mathbf{e}_y$ using the energy principle for $\partial\boldsymbol{\xi}/\partial y = 0$. (the perturbation on the boundary is 0)

Hint: Compute the terms $\nabla\cdot\boldsymbol{\xi}$, $\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0)$, $\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0)$ and by using the equilibrium condition $\nabla\left(p + B^2/2\mu_0\right) = 0$ try to bring the potential into the form

$$U = \frac{1}{2}\int_V \left[a_{11}\left(\nabla\cdot\boldsymbol{\xi}\right)^2 + 2a_{12}\xi_x\nabla\cdot\boldsymbol{\xi} + a_{22}\xi_x^2\right] d\mathbf{x}$$

Stability is then obtained for the condition: $a_{11}a_{22} - a_{12}^2 > 0$. Evaluate this condition for the Harris sheet.

Please turn in the solutions to the homework on Wednesday, 11/22/2006