Plasma Physics - Homework 9, 11/17/2006

1. Force balance equation in cylindrical coordinates:

Assume $\partial/\partial\theta = \partial/\partial z$ and $B_r = 0$ and show that the force balance equation in cylindrical coordinates (r, θ, z) takes the form

$$\frac{\partial}{\partial r}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_0}\right) + \frac{B_\theta^2}{\mu_0 r} = 0$$

Force balance equation:

$$-\nabla p + \mathbf{j} \times \mathbf{B} = 0$$

in cylindrical coordinates with $\partial/\partial\theta = \partial/\partial z$

$$\begin{aligned} \nabla p &= \frac{\partial p}{\partial r}\mathbf{e}_r \\ \mu_0 \mathbf{j} &= -\frac{\partial B_z}{\partial r}\mathbf{e}_\theta + \frac{1}{r}\frac{\partial r B_\theta}{\partial r}\mathbf{e}_z \end{aligned}$$

such that

$$\begin{aligned} \mathbf{j} \times \mathbf{B} &= \tfrac{1}{\mu_0} \left(-\frac{\partial B_z}{\partial r}B_z - \frac{1}{r}\frac{\partial r B_\theta}{\partial r}B_\theta\right)\mathbf{e}_r \\ &= -\frac{1}{2\mu_0}\frac{d}{dr}\left(B_z^2 + B_\theta^2\right) - \frac{B_\theta^2}{\mu_0} \end{aligned}$$

yielding the force balance equation

$$\frac{dp}{dr} + \frac{1}{2\mu_0}\frac{d}{dr}\left(B_z^2 + B_\theta^2\right) + \frac{B_\theta^2}{\mu_0} = 0$$

2. θ pinch:

Constant current j_0 in the z direction in a cylindrical coordinate system.

Magnetic field $B_\theta(r)$:

$$\begin{aligned} \frac{1}{r}\frac{\partial r B_\theta}{\partial r} &= \mu_0 j_0 \\ B_\theta &= \frac{\mu_0 j_0}{r}\int^r r dr = \frac{\mu_0 j_0}{2} r \end{aligned}$$

Note $B_\theta = 0$ for $r = 0$ because $r = 0$ is a singular point for B_θ.

Pressure:

$$\frac{dp}{dr} = -j_z B_\theta$$

which yields

$$p = -\frac{\mu_0 j_0^2}{2}\int^r r dr = -\frac{\mu_0 j_0^2}{4} r^2 + p_0$$

such that $p(0) = p_0$. The pressure assumes 0 for

$$-\frac{\mu_0 j_0^2}{4} r_c^2 + p_0 = 0$$

or

$$r_c^2 = \frac{4 p_0}{\mu_0 j_0^2}$$

Stability criterion:

$$-\frac{r}{p}\frac{dp}{dr} < \frac{4\gamma}{2 + \gamma\beta}$$

for stability. With

$$\begin{aligned} -\frac{r}{p}\frac{dp}{dr} &= \frac{r}{p_0 - \frac{\mu_0 j_0^2}{4} r^2}\frac{\mu_0 j_0^2}{2} r \\ &= \frac{2 r^2 / r_c^2}{(1 - r^2/r_c^2)} \end{aligned}$$

and

$$\begin{aligned}\beta = \frac{2\mu_0 p}{B^2} &= \frac{2\mu_0\left(p_0 - \frac{\mu_0 j_0^2}{4} r^2\right)}{\frac{\mu_0^2 j_0^2}{4} r^2} \\ &= \frac{2\left(1 - r^2/r_c^2\right)}{r^2/r_c^2}\end{aligned}$$

Substitution in the stability condition:

$$\frac{2r^2/r_c^2}{\left(1 - r^2/r_c^2\right)} < \frac{4\gamma}{2 + \gamma\beta} = \frac{2\gamma}{1 + \gamma\frac{\left(1 - r^2/r_c^2\right)}{r^2/r_c^2}} = \frac{2\gamma r^2/r_c^2}{r^2/r_c^2 + \gamma\left(1 - r^2/r_c^2\right)}$$

or

$$r^2/r_c^2 < 0$$

Thus this configuration is everywhere unstable. Note that this is caused by the assumed current density profile.

Often β is assumed to be constant or small. Assuming that β is constant the stability condition is

$$-\frac{r}{p}\frac{dp}{dr} = \frac{2r^2/r_c^2}{\left(1 - r^2/r_c^2\right)} < \frac{4\gamma}{2 + \gamma\beta} = \lambda$$

or

$$\begin{aligned} r^2 &< \frac{\lambda}{2 + \lambda} r_c^2 \\ &= \frac{\gamma}{\gamma + 1} r_c^2 \qquad \text{for } \beta \ll 1\end{aligned}$$

for stability.

Thus the small β approximation yields the correct result for for sufficiently large radii but fails for smaller values of r/r_c.

3. Stability for the Harris sheet:

Determine the stability criterion for the Harris sheet equilibrium using the energy principle. Assume that the perturbation on the boundary is 0

Harris sheet with

$$\begin{aligned}
\mathbf{B} = B\mathbf{e}_y &= B_0 \tanh\frac{x}{L}\,\mathbf{e}_y \\
p &= p_0 \cosh^{-2}\frac{x}{L} \\
\mathbf{j} &= \frac{B_0}{\mu_0 L}\cosh^{-2}\frac{x}{L}\,\mathbf{e}_z
\end{aligned}$$

The energy principle for a displacement $\boldsymbol{\xi}$ without considering surface terms is

$$\begin{aligned}
U &= \frac{1}{2}\int_V \left[\gamma p_0 \left(\nabla\cdot\boldsymbol{\xi}\right)^2 + \frac{1}{\mu_0}\left(\nabla\times\left(\boldsymbol{\xi}\times\mathbf{B}_0\right)\right)^2 \right. \\
&\quad \left. +\boldsymbol{\xi}\cdot\nabla p_0 \nabla\cdot\boldsymbol{\xi} - \frac{1}{\mu_0}\left(\boldsymbol{\xi}\times\left(\nabla\times\mathbf{B}_0\right)\right)\cdot\nabla\times\left(\boldsymbol{\xi}\times\mathbf{B}_0\right)\right] d\mathbf{x}
\end{aligned}$$

With

$$\begin{aligned}
\nabla\cdot\boldsymbol{\xi} &= \frac{\partial\xi_x}{\partial x} + \frac{\partial\xi_y}{\partial y} + \frac{\partial\xi_z}{\partial z} \\
\nabla\times\left(\boldsymbol{\xi}\times\mathbf{B}_0\right) &= B\left[\frac{\partial\xi_x}{\partial y}\mathbf{e}_x - \left(\frac{\partial\xi_z}{\partial z} + \frac{1}{B}\frac{\partial\xi_x B}{\partial x}\right)\mathbf{e}_y + \frac{\partial\xi_z}{\partial y}\mathbf{e}_z\right] \\
\boldsymbol{\xi}\times\left(\nabla\times\mathbf{B}_0\right) &= \frac{\partial B}{\partial x}\left(\xi_y\mathbf{e}_x - \xi_x\mathbf{e}_y\right)
\end{aligned}$$

such that the potential becomes

$$\begin{aligned}
U &= \frac{1}{2}\int_V \left[\gamma p\left(\nabla\cdot\boldsymbol{\xi}\right)^2 + \frac{\partial p}{\partial x}\xi_x\nabla\cdot\boldsymbol{\xi} \right. \\
&\quad +\frac{B^2}{\mu_0}\left[\left(\frac{\partial\xi_x}{\partial y}\right)^2 + \left(\frac{\partial\xi_z}{\partial z} + \frac{1}{B}\frac{\partial\xi_x B}{\partial x}\right)^2 + \left(\frac{\partial\xi_z}{\partial y}\right)^2\right] \\
&\quad \left. -\frac{1}{2\mu_0}\frac{\partial B^2}{\partial x}\left(\xi_y\frac{\partial\xi_x}{\partial y} + \xi_x\left(\frac{\partial\xi_z}{\partial z} + \frac{1}{B}\frac{\partial\xi_x B}{\partial x}\right)\right)\right] d\mathbf{x}
\end{aligned}$$

For $\partial/\partial y = 0$ and $\nabla\left(p + B^2/2\mu_0\right) = 0$:

$$\begin{aligned}
U &= \frac{1}{2}\int_V \left[\gamma p\left(\nabla\cdot\boldsymbol{\xi}\right)^2 + \frac{\partial p}{\partial x}\xi_x\nabla\cdot\boldsymbol{\xi} + \frac{B^2}{\mu_0}\left(\frac{\partial\xi_z}{\partial z} + \frac{1}{B}\frac{\partial\xi_x B}{\partial x}\right)^2 \right. \\
&\quad \left. +\frac{\partial p}{\partial x}\xi_x\left(\frac{\partial\xi_z}{\partial z} + \frac{1}{B}\frac{\partial\xi_x B}{\partial x}\right)\right] d\mathbf{x}
\end{aligned}$$

With

$$\frac{\partial \xi_z}{\partial z} + \frac{1}{B}\frac{\partial \xi_x B}{\partial x} = \nabla \cdot \boldsymbol{\xi} + \xi_x \frac{1}{B}\frac{dB}{dx}$$

we obtain

$$\begin{aligned} U &= \frac{1}{2}\int_V \left[\gamma p_0 \left(\nabla \cdot \boldsymbol{\xi}\right)^2 + \frac{\partial p}{\partial x}\xi_x \nabla \cdot \boldsymbol{\xi} + \frac{B^2}{\mu_0}\left(\nabla \cdot \boldsymbol{\xi} + \xi_x \frac{1}{B}\frac{dB}{dx}\right)^2 \right. \\ &\qquad \left. + \frac{\partial p}{\partial x}\xi_x \left(\nabla \cdot \boldsymbol{\xi} + \xi_x \frac{1}{B}\frac{dB}{dx}\right)\right] d\mathbf{x} \\ &= \frac{1}{2}\int_V \left[a_{11}\left(\nabla \cdot \boldsymbol{\xi}\right)^2 + 2a_{12}\xi_x \nabla \cdot \boldsymbol{\xi} + a_{22}\xi_x^2\right] d\mathbf{x} \end{aligned}$$

with

$$\begin{aligned} a_{11} &= \gamma p_0 + \frac{B^2}{\mu_0} \\ a_{12} &= \frac{dp}{dx} + \frac{B}{\mu_0}\frac{dB}{dx} \\ a_{22} &= \frac{1}{\mu_0}\left(\frac{dB}{dx}\right)^2 + \frac{\partial p}{\partial x}\frac{1}{B}\frac{dB}{dx} \end{aligned}$$

Using the equilibrium condition for the Harris sheet:

$$0 = \frac{d}{dx}\left(p + \frac{B^2}{2\mu_0}\right) = \frac{dp}{dx} + \frac{B}{\mu_0}\frac{dB}{dx}$$

we obtain $a_{12} = 0$ and

$$a_{22} = \frac{1}{\mu_0}\frac{1}{B}\frac{dB}{dx}\left(B\frac{dB}{dx}\right) + \frac{\partial p}{\partial x}\frac{1}{B}\frac{dB}{dx} = 0$$

such that the equilibrium condition reads

$$a_{11}a_{22} - a_{12}^2 = 0 \geq 0$$

This is insufficient to clarify stability uniquely, but:

Compressible perturbations are always stable because $a_{11} > 0$. For incompressible perturbations the result is inconclusive.